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Properties of convergence for the q-Meyer-König and Zeller operators ☆

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Abstract

In this paper, we discuss properties of convergence for the q-Meyer-König and Zeller operators $M_{n,q}$. Based on an explicit expression for $M_{n,q}(t^2,x)$ in terms of q-hypergeometric series, we show that for $q_n \in (0,1]$, the sequence $(M_{n,q_n}(f))_{n\geqslant 1}$ converges to f uniformly on $[0,1]$ for each $f \in C[0,1]$ if and only if $\lim_{n\to\infty} q_n = 1$. For fixed $q \in (0,1)$, we prove that the sequence $(M_{n,q}(f))$ converges for each $f \in C[0,1]$ and obtain the estimates for the rate of convergence of $(M_{n,q}(f))$ by the modulus of continuity of f, and the estimates are sharp in the sense of order for Lipschitz continuous functions. We also give explicit formulas of Voronovskaya type for the q-Meyer-König and Zeller operators for fixed $0 < q < 1$. If $0 < q < 1$, $f \in C^1[0,1]$, we show that the rate of convergence for the Meyer-König and Zeller operators is $o(q^n)$ if and only if

$$\frac{f(1-q^{k-1}) - f(1-q^k)}{(1-q^{k-1}) - (1-q^k)} = f'\left(1-q^k\right), \quad k = 1, 2, \ldots.$$

Keywords: q-Meyer-König and Zeller operators; Rate of approximation; Modulus of smoothness; Voronovskaya type formulas

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1. Introduction

Let $q \in (0, 1]$ (throughout the paper we always assume $q \in (0, 1]$). For each nonnegative integer k, the q-integer $[k]$ and the q-factorial $[k]!$ are defined by

$$[k] := [k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1, \end{cases}$$

and

$$[k]! := \begin{cases} [k][k-1]\cdots[1], & k \geqslant 1, \\ 1, & k = 0, \end{cases}$$

respectively. For the integers n, k, $n \geqslant k \geqslant 0$, the q-binomial, or the Gaussian coefficient is defined by (see [5, p. 12])

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]![n-k]!}.$$

We also use the following standard notations (see [3, pp. 3, 6]):

$$(a;q)_0 := 1, \qquad (a;q)_k := \prod_{s=0}^{k-1} \left(1 - aq^s\right), \qquad (a;q)_\infty := \prod_{s=0}^{\infty} \left(1 - aq^s\right).$$

Clearly,

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

In [9], Phillips proposed the q-Bernstein polynomials: for each positive integer n, and $f \in C[0, 1]$, the q-Bernstein polynomial of f is

$$B_{n,q}(f, x) := \sum_{k=0}^{n} f\left(\frac{[k]}{[n]}\right) \begin{bmatrix} n \\ k \end{bmatrix} x^k (x;q)_{n-k}. \tag{1.1}$$

Note that for $q = 1$, $B_{n,q}(f, x)$ is the classical Bernstein polynomial $B_n(f, x)$:

$$B_n(f, x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}.$$

In [12], Tiberiu Trif introduced the q-Meyer-König and Zeller operators (or the q-MKZ operators for simplicity): for each positive integer n, and $f \in C[0, 1]$,

$$M_{n,q}(f, x) := \begin{cases} \sum_{k=0}^{\infty} f\left(\frac{[k]}{[n+k]}\right) \begin{bmatrix} n+k \\ k \end{bmatrix} x^k (x;q)_{n+1}, & 0 \leqslant x < 1, \\ f(1), & x = 1. \end{cases} \tag{1.2}$$

When $q = 1$, the q-MKZ operator $M_{n,q}$ reduces to the classical Meyer-König and Zeller operator M_n:

$$M_n(f, x) := \begin{cases} \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) \binom{n+k}{k} x^k (1-x)^{n+1}, & 0 \leqslant x < 1, \\ f(1), & x = 1. \end{cases}$$

The q-Bernstein polynomials and the q-MKZ operators share good properties such as the shape-preserving properties and monotonicity from above for convex function (see [10, pp. 270, 280–290], [12]). It is well known that for $q \neq 1$, convergence properties of the q-Bernstein polynomials are not similar to those of the classical ones. While the q-Bernstein polynomials have been studied widely by a number of authors (see [4,6,8–10] and references therein, [11,13–15]), there are very few works about the q-MKZ operators as far as we know. In this paper, we shall study properties of convergence of the q-MKZ operators. Our results demonstrate that in general properties of convergence for the q-MKZ operators are essentially different from those for the classical MKZ operators, however they are very similar to those for the q-Bernstein polynomials.

Throughout the paper, we always assume that f is a continuous real function on $[0,1]$. The expression $g_n(x) \rightrightarrows g(x)$ $[x \in [0,1]; n \to \infty]$ denotes convergence of g_n to g uniformly in $x \in [0,1]$ as $n \to \infty$; $A(n) \asymp B(n)$ means that $A(n) \ll B(n)$ and $A(n) \gg B(n)$, and $A(n) \ll B(n)$ means that there exists a positive constant c independent of n such that $A(n) \leqslant cB(n)$; $A(n) = o(B(n))$ represents $\lim_{n\to\infty} A(n)/B(n) = 0$.

2. Statement of results

In [12], Tiberiu Trif investigated approximating properties of the q-MKZ operators $M_{n,q_n}(f,x)$ and obtained the following results:

Theorem A. *If $(q_n)_{n\geqslant 1}$ is a sequence of real numbers satisfying $1 - 1/n \leqslant q_n \leqslant 1$, then for any $f \in C[0,1]$, the sequence $(M_{n,q_n}(f))_{n\geqslant 1}$ converges to f uniformly on $[0,1]$.*

In Section 3, we shall discuss further approximating properties of the q-MKZ operators. From the definition of the q-MKZ operators $M_{n,q}$ we know that $M_{n,q}$ are positive linear operators. Hence the moments $M_{n,q}(t^r,x)$ $(r = 0,1,2)$ are of particular importance by the theory of approximation by positive operators (see [2, pp. 277–281]). It was proved in [12] that $M_{n,q}(f,x)$ reproduce linear functions, in other words,

$$M_{n,q}(1,x) = 1 \quad \text{and} \quad M_{n,q}(t,x) = x. \tag{2.1}$$

So we need to compute $M_{n,q}(t^2,x)$. In the case $q = 1$, Alkemade [1] first derived an explicit expression for $M_n(t^2,x)$ in terms of hypergeometric series (see the definition in Section 3):

$$M_n\big(t^2,x\big) = x^2 + \frac{x(1-x)^2}{n+1} {}_2F_1(1,2;n+2;x) \quad \big(x \in [0,1)\big).$$

In Section 3, we shall give an explicit expression for $M_{n,q}(t^2,x)$ in terms of q-hypergeometric series. Possibly our formula is new even for the classical MKZ operators.

Theorem 1. *For $x \in [0,1)$,*

$$M_{n,q}\big(t^2,x\big) = \begin{cases} x^2 + \frac{x(1-x)}{[n+1]}\big(1 - \frac{q^{n+2}[n]x}{[n+2]} {}_2\phi_1(q,q^2;q^{n+3};q,q^{n+1}x)\big), & q < 1, \\ x^2 + \frac{x(1-x)}{n+1}\big(1 - \frac{nx}{n+2} {}_2F_1(1,2;n+3;x)\big), & q = 1, \end{cases} \tag{2.2}$$

where the definition of the q-hypergeometric series is given in Section 3.

Based on Theorem 1, we also have the following approximation theorem.

Theorem 2. *Let $q_n \in (0,1]$. Then the sequence $(M_{n,q_n}(f))_{n\geqslant 1}$ converges to f uniformly on $[0,1]$ for each $f \in C[0,1]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Theorem 1 implies that if $q \in (0,1)$ is fixed, $(M_{n,q}(f,x))$ may not be approximating for some continuous functions. In Section 4, we shall discuss properties of convergence for the q-MKZ operators for fixed q, $0 < q < 1$. It was proved in [4] that for each $f \in C[0,1]$, the sequence $(B_{n,q}(f,x))$ converges to $B_{\infty,q}(f,x)$ as $n \to \infty$ uniformly in $x \in [0,1]$ and $q \in (0,1]$, where $B_{\infty,1}(f) = f$ and for $0 < q < 1$,

$$B_{\infty,q}(f,x) := \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)\frac{x^k}{(q;q)_k}(x;q)_\infty, & 0 \leqslant x < 1, \\ f(1), & x = 1. \end{cases} \tag{2.3}$$

For results about properties of $B_{\infty,q}(f,x)$ we refer to [4,7,8,11]. For $f \in C[0,1]$, $t > 0$, we define the modulus of continuity $\omega(f,t)$ and the second modulus of smoothness $\omega_2(f,t)$ as follows:

$$\omega(f;t) := \sup_{\substack{|x-y|\leqslant t \\ x,y\in[0,1]}} \bigl|f(x) - f(y)\bigr|;$$

$$\omega_2(f,t) := \sup_{0<h\leqslant t} \sup_{x\in[0,1-2h]} \bigl|f(x+2h) - 2f(x+h) + f(x)\bigr|.$$

Theorem 3. *Let $q \in (0,1)$, and let $f \in C[0,1]$. Then*

$$\bigl\|M_{n,q}(f) - B_{\infty,q}(f)\bigr\| \leqslant C_q\omega\bigl(f,q^n\bigr). \tag{2.4}$$

The above estimate is sharp in the following sense of order: for each α, $0 < \alpha \leqslant 1$, there exists a function $f_\alpha(x)$ which belongs to the Lipschitz class $\mathrm{Lip}\,\alpha := \{f \in C[0,1] \mid \omega(f,t) \ll t^\alpha\}$ such that

$$\bigl\|M_{n,q}(f_\alpha) - B_{\infty,q}(f_\alpha)\bigr\| \gg q^{\alpha n}. \tag{2.5}$$

Theorem 4. *Let $0 < q < 1$. Then*

$$\bigl\|M_{n,q}(f) - B_{\infty,q}(f)\bigr\| \leqslant c\omega_2\bigl(f,\sqrt{q^n}\bigr). \tag{2.6}$$

Furthermore,

$$\sup_{q\in(0,1]} \bigl\|M_{n,q}(f) - B_{\infty,q}(f)\bigr\| \leqslant c\omega_2\bigl(f,n^{-1/2}\bigr), \tag{2.7}$$

where c is an absolute constant.

Remark 1. Theorem 4 is announced in [13] without proof. From (2.7) we know that for each $f \in C[0,1]$, $\lim_{n\to\infty} M_{n,q}(f,x) = B_{\infty,q}(f,x)$ uniformly in $x \in [0,1]$ and $q \in (0,1]$. Since the equality $B_{\infty,q}(f,x) = f(x)$ holds if and only if f is linear (see [4]), we know for $q \in (0,1)$ and $f \in C[0,1]$, the sequence $(M_{n,q}(f,x))$ does not approximate $f(x)$ unless f is linear. This is completely in contrast to the case $q = 1$, where $(M_n(f,x))$ approximates $f(x)$ for any $f \in C[0,1]$.

Remark 2. Results similar to Theorem 3 for the q-Bernstein polynomials were obtained in [14]. Note that when $f(x) = x^2$, we have (see Theorem 5)

$$\bigl\|M_{n,q}(f) - B_{\infty,q}(f)\bigr\| \asymp q^n \asymp \omega_2\bigl(f,\sqrt{q^n}\bigr).$$

Hence, the estimate (2.6) is sharp in the following sense: the sequence $\sqrt{q^n}$ in (2.6) cannot be replaced by any other sequence decreasing to zero more rapidly as $n \to \infty$. However, (2.6) is not sharp for the Lipschitz class $\mathrm{Lip}\,\alpha$ ($\alpha \in (0,1]$) in the sense of order. This, combining with Theorem 3, shows that in the case $0 < q < 1$, the modulus of continuity is more appropriate to describe the rate of convergence for the q-MKZ operators than the second modulus of smoothness.

Remark 3. In the case $0 < q < 1$, from (2.4) we conclude that the rate of convergence $\|M_{n,q}(f) - B_{\infty,q}(f)\|$ has the order q^n for each $f \in C^1[0,1]$ versus at most $1/n$ for the classical MKZ operators. From (2.7) we know that the rate of convergence $\|M_{n,q}(f) - B_{\infty,q}(f)\|$ can be dominated by $c\omega_2(f, n^{-1/2})$ uniformly with respect to $q \in (0,1]$.

Remark 4. The constant c in (2.6) is an absolute constant and does not depend on q, however the constant C_q in (2.4) depends on q, and tends to $+\infty$ as $q \to 1-$. Hence, (2.6) does not follow from (2.4).

In Section 5, we study Voronovskaya type formulas for the q-MKZ operators for fixed $q \in (0,1)$. In the case $q = 1$, we know that for any $f \in C^2[0,1]$,

$$\lim_{n\to\infty} n\big(M_n(f,x) - f(x)\big) = f''(x)x(1-x)^2/2 =: V_1(f,x) \tag{2.8}$$

uniform in $x \in [0,1]$. Hence, the classical MKZ operators $M_n(f,x)$ possess saturation: no function $f \in C[0,1]$ can be approximated with error better than $o(1/n)$ unless it is linear. In the case $0 < q < 1$, for the q-Bernstein polynomials, it was proved in [15] that for any $f \in C^1[0,1]$,

$$\lim_{n\to\infty} \frac{[n]}{q^n}\big(B_{n,q}(f,x) - B_{\infty,q}(f,x)\big) = L_q(f,x)$$

uniformly in $x \in [0,1]$, where

$$L_q(f,x) := \begin{cases} \sum_{k=1}^{\infty}[k]\big(f'(1-q^k) - \frac{f(1-q^k)-f(1-q^{k-1})}{(1-q^k)-(1-q^{k-1})}\big)\frac{x^k}{(q;q)_k}(x;q)_\infty, & x \in [0,1), \\ 0, & x = 1. \end{cases} \tag{2.9}$$

Similarly, we have the following Voronovskaya type theorem for the q-MKZ operators for fixed $q \in (0,1)$.

Theorem 5. *Let* $0 < q < 1$, $f \in C^1[0,1]$. *Then*

$$\lim_{n\to\infty} \frac{[n]}{q^n}\big(M_{n,q}(f,x) - B_{\infty,q}(f,x)\big) = V_q(f,x) \tag{2.10}$$

uniformly in $x \in [0,1]$, *where*

$$V_q(f,x) := \begin{cases} \sum_{k=1}^{\infty}[k]\big(f'(1-q^k) - \frac{f(1-q^k)-f(1-q^{k-1})}{(1-q^k)-(1-q^{k-1})}\big)\frac{q^k x^k}{(q;q)_k}(x;q)_\infty, & x \in [0,1), \\ 0, & x = 1. \end{cases} \tag{2.11}$$

From Theorem 5, we have the following saturation of convergence for the q-MKZ operators for fixed $q \in (0,1)$.

Corollary 1. *Let $0 < q < 1$ and $f \in C^1[0,1]$. Then $\|M_{n,q}(f) - B_{\infty,q}(f)\| = o(q^n)$ if and only if $V_q(f,x) \equiv 0$, and this is equivalent to*

$$\frac{f(1-q^{k-1}) - f(1-q^k)}{(1-q^{k-1}) - (1-q^k)} = f'\left(1-q^k\right), \quad k = 1, 2, \ldots. \tag{2.12}$$

Remark 5. It is easy to see that $V_q(f,x) = (1-x)L_q(f,qx)$. Using the same method as in the proof of Theorem 2 in [15], we can prove that for any $f \in C^2[0,1]$,

$$\lim_{q \to 1-} V_q(f,x) = V_1(f,x)$$

uniformly in $x \in [0,1]$.

Remark 6. It can be readily seen from (2.12) that for fixed $q \in (0,1)$, there exist numerous nonlinear continuously differentiable functions f such that $V_q(f,x) \equiv 0$. However, if we assume that the function f is convex on $[0,1]$ or analytic on $(-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$, then the rate of convergence for the q-MKZ operators is $o(q^n)$ if and only if f is linear (see [15]).

Remark 7. The above theorems are also true for complex-valued functions.

3. Proofs of Theorems 1–2

We introduce some notations. For $0 < q < 1$, denote the q-derivative $D_q f(x)$ of f by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

The hypergeometric series and the q-hypergeometric series are defined by

$${}_2F_1(a,b;c;z) = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} z^k,$$

and

$${}_2\phi_1(a,b;c;q,z) = {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(q;q)_k (c;q)_k} z^k, \tag{3.1}$$

respectively, where $(a)_k$ is shifted factorial defined by

$$(a)_0 = 1, \qquad (a)_k = a(a+1)\cdots(a+k-1), \quad k = 1, 2, \ldots.$$

Proof of Theorem 1. We only give the proof of the case $0 < q < 1$, the proof of the case $q = 1$ is similar. For $x \in (0,1)$, direct computation gives that (see [12])

$$M_{n,q}\left(t^2, x\right) - x^2 = x(x;q)_{n+1} \sum_{k=0}^{\infty} \frac{(qx)^k}{[n+k+1]} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}.$$

Denote

$$Z(qx) = \frac{M_{n,q}(t^2,x) - x^2}{x(1-x)}. \tag{3.2}$$

Then

$$Z(x)=(x;q)_n\sum_{k=0}^{\infty}\frac{x^k}{[n+k+1]}\begin{bmatrix}n+k-1\\k\end{bmatrix}.$$

By simple computation, using the Heine binomial formulas (see [5, p. 28])

$$\frac{1}{(x;q)_n}=\sum_{k=0}^{\infty}\begin{bmatrix}n+k-1\\k\end{bmatrix}x^k,$$

we get

$$x\bigl(1-q^nx\bigr)D_qZ(x)=(qx;q)_n\sum_{k=0}^{\infty}\frac{x^k}{[n+k+1]}\begin{bmatrix}n+k-1\\k\end{bmatrix}\bigl([k]-[n+k]x\bigr),$$

$$Z(qx)\bigl([n+1]-q^nx\bigr)=(qx;q)_n\sum_{k=0}^{\infty}\frac{x^k}{[n+k+1]}\begin{bmatrix}n+k-1\\k\end{bmatrix}\bigl(q^k[n+1]-q^{n+k}x\bigr),$$

and

$$x\bigl(1-q^nx\bigr)D_qZ(x)+Z(qx)\bigl([n+1]-q^nx\bigr)=(x;q)_{n+1}\sum_{k=0}^{\infty}\begin{bmatrix}n+k-1\\k\end{bmatrix}x^k=1-q^nx.$$

We assume that $Z(x)=\sum_{k=0}^{\infty}c_kx^k$. Then

$$x\bigl(1-q^nx\bigr)D_qZ(x)=\sum_{k=1}^{\infty}\bigl(c_k[k]-q^nc_{k-1}[k-1]\bigr)x^k,$$

$$Z(qx)\bigl([n+1]-q^nx\bigr)=[n+1]c_0+\sum_{k=1}^{\infty}\bigl(q^kc_k[n+1]-q^{n+k-1}c_{k-1}\bigr)x^k,$$

and therefore,

$$\begin{aligned}&x\bigl(1-q^nx\bigr)D_qZ(x)+Z(qx)\bigl([n+1]-q^nx\bigr)\\&\quad=[n+1]c_0+\sum_{k=1}^{\infty}\bigl(c_k[n+k+1]-q^nc_{k-1}[k]\bigr)x^k=1-q^nx.\end{aligned}\tag{3.3}$$

From (3.3) we get

$$[n+1]c_0=1;\qquad c_1[n+2]-q^nc_0=-q^n;$$
$$c_k[n+k+1]-q^nc_{k-1}[k]=0\quad(k\geqslant 2).$$

Then

$$c_0=\frac{1}{[n+1]},\qquad c_1=-\frac{q^{n+1}[n]}{[n+2][n+1]},$$
$$c_k=q^{n(k-1)}\frac{[k][k-1]\cdots[2]}{[n+k+1][n+k]\cdots[n+3]}c_1,$$

where

$$\frac{[k][k-1]\cdots[2]}{[n+k+1][n+k]\cdots[n+3]}=\frac{(1-q^k)(1-q^{k-1})\cdots(1-q^2)}{(1-q^{n+k+2})\cdots(1-q^{n+3})}=\frac{(q^2;q)_{k-1}}{(q^{n+3};q)_{k-1}}.$$

Hence, (2.2) can be inferred from (3.2) and the following relations:

$$Z(x) = c_0 + x\sum_{k=0}^{\infty} c_{k+1}x^k = c_0 + c_1 x\sum_{k=0}^{\infty}\frac{(q^2;q)_k}{(q^{n+3};q)_k}\left(q^n x\right)^k$$
$$= \frac{1}{[n+1]}\left(1 - \frac{q^{n+1}[n]x}{[n+2]}{}_2\phi_1\left(q,q^2;q^{n+3};q,q^n x\right)\right).$$

Theorem 1 is proved. □

Proof of Theorem 2. Since the q-MKZ operators M_{n,q_n} are positive linear operators and reproduce linear functions, the well-known Korovkin theorem implies that $M_{n,q_n}(f,x) \rightrightarrows f(x)$ $[x \in [0,1]; n \to \infty]$ for any $f \in C[0,1]$ if and only if

$$M_{n,q_n}\left(t^2,x\right) \rightrightarrows x^2 \quad \left[x \in [0,1]; n \to \infty\right]. \tag{3.4}$$

So we need to estimate the quantity $M_{n,q_n}(t^2,x) - x^2$. From [2, p. 281] we know that if a positive linear operator L on $C[0,1]$ reproduces linear functions, then $L(f,x) \geqslant f(x)$ for any convex function f and for any $x \in [0,1]$. By (2.2) and the fact that

$${}_2\phi_1\left(q,q^2;q^{n+3};q,q^{n+1}x\right) \geqslant 0, \quad {}_2\phi_1(1,2;n+3;x) \geqslant 0 \quad \left(x \in [0,1)\right),$$

we get

$$0 \leqslant M_{n,q}\left(t^2,x\right) - x^2 \leqslant \frac{x(1-x)}{[n+1]}. \tag{3.5}$$

Since $\sup_{0<q<1}\frac{q^{n+3}(1-q)}{1-q^{n+3}} = \frac{1}{n+3}$, we have for $q \in (0,1)$,

$$1 - \frac{q^{n+2}[n]}{2[n+2]}{}_2\phi_1\left(q,q^2;q^{n+3};q,q^{n+1}/2\right)$$
$$= 1 - \frac{q^{n+2}[n]}{2[n+2]}\sum_{s=0}^{\infty}\frac{(q^2;q)_s}{(q^{n+3};q)_s}\left(q^{n+1}/2\right)^s$$
$$\geqslant 1 - \frac{q^{n+2}}{2}\left(1 + \frac{(1-q^2)q^{n+1}}{2(1-q^{n+3})}\sum_{s=1}^{\infty}\left(q^{n+1}/2\right)^{s-1}\right) \geqslant 1 - 1/2 - \frac{q^{n+2}}{2}\frac{(1-q^2)q^{n+1}}{(1-q^{n+3})}$$
$$\geqslant 1/2 - \frac{q^{n+3}(1-q)}{1-q^{n+3}}\frac{1+q}{2} \geqslant 1/2 - \frac{1}{n+3} \geqslant 1/4,$$

and hence,

$$M_{n,q}\left(t^2,1/2\right) - 1/4 \geqslant \frac{1}{16[n+1]}. \tag{3.6}$$

Now we suppose that $q_n \to 1$. Then $\lim_{n\to\infty}[n+1]_{q_n} = \infty$ (see [11]), and (3.4) follows from (3.5). Hence, if $q_n \to 1$, then for any $f \in C[0,1]$,

$$M_{n,q_n}(f,x) \rightrightarrows f(x) \quad \left[x \in [0,1]; n \to \infty\right].$$

On the other hand, if we assume that for any $f \in C[0,1]$, $M_{n,q_n}(f,x) \rightrightarrows f(x)$ $[x \in [0,1];$ $n \to \infty]$, then $q_n \to 1$. In fact, if the sequence (q_n) does not tend to 1, then it must contain a subsequence (q_{n_k}) such that $q_{n_k} \in (0,1)$, $q_{n_k} \to t \in [0,1)$ as $k \to \infty$. Thus, $\frac{1}{[n_k+1]_{q_{n_k}}} = \frac{1-q_{n_k}}{1-(q_{n_k})^{n_k+1}} \to (1-t)$ as $k \to \infty$. Taking $x = 1/2$, $n = n_k$, $q = q_{n_k}$ in (3.4), we get

$$M_{n_k,q_{n_k}}\bigl(t^2, 1/2\bigr) - 1/4 \to 0 \quad (k \to \infty).$$

However, by (3.6) we have

$$M_{n_k,q_{n_k}}\bigl(t^2, 1/2\bigr) - 1/4 \geqslant \frac{1}{16[n_k+1]_{q_{n_k}}} \to \frac{1-t}{16} > 0.$$

This leads to a contradiction. Hence, $q_n \to 1$. Theorem 2 is proved. □

4. Proofs of Theorems 3–4

For integers n, k and $q \in (0,1)$, $x \in [0,1]$, we set

$$m_{nk}(q;x) := \begin{bmatrix} n+k \\ k \end{bmatrix} x^k (x;q)_{n+1}, \qquad m_{\infty k}(q;x) := \frac{x^k}{(q;q)_k}(x;q)_\infty.$$

Then

$$\begin{aligned} m_{nk}(q;x) - m_{\infty k}(q;x) &= \left(\left(\prod_{s=n+1}^{\infty}\bigl(1-q^s x\bigr)\right)^{-1} \prod_{s=n+1}^{n+k}\bigl(1-q^s\bigr) - 1\right) m_{\infty k}(q;x) \\ &= \bigl(e^J - 1\bigr) m_{\infty k}(q;x), \end{aligned} \tag{4.1}$$

where $J := -\sum_{s=n+1}^{\infty} \ln(1-q^s x) + \sum_{s=n+1}^{n+k} \ln(1-q^s)$. We first prove the following lemma.

Lemma 1. *Let* $0 < q < 1$. *Then for integers* n, k *and for* $x \in [0,1]$,

$$\bigl|m_{nk}(q;x) - m_{\infty k}(q;x)\bigr| \leqslant c q^n m_{\infty k}(q;x) \tag{4.2}$$

and

$$\left|\frac{[n]}{q^n}\bigl(m_{nk}(q;x) - m_{\infty k}(q;x)\bigr) - \left(\frac{qx}{(1-q)^2} - \frac{q[k]}{1-q}\right) m_{\infty k}(q;x)\right| \leqslant c q^n m_{\infty k}(q;x), \tag{4.3}$$

where the constants in (4.2) *and* (4.3) *depend only on* q.

Proof. It suffices to prove (4.3), since (4.2) follows from (4.3). From (4.1) we know that (4.3) is equivalent to

$$K := \left|q^{-n}\bigl(1-q^n\bigr)\bigl(e^J - 1\bigr) - \frac{qx}{1-q} + q[k]\right| \leqslant c q^n. \tag{4.4}$$

First we estimate J. For $t \in (0,q]$, we have

$$0 \leqslant \frac{1}{t}\ln\frac{1}{1-t} - 1 = \sum_{n=1}^{\infty} \frac{t^n}{n+1} \leqslant \frac{t}{2}\sum_{n=1}^{\infty} q^{n-1} = \frac{t}{2(1-q)}.$$

It follows that

$$-\frac{t^2}{2(1-q)} \leqslant \ln(1-t) + t \leqslant 0,$$

and therefore, for $x \in [0,1]$,

$$-\sum_{s=n+1}^{n+k} \frac{q^{2s}}{2(1-q)} \leqslant \sum_{s=n+1}^{n+k} \ln\bigl(1-q^s\bigr) + \sum_{s=n+1}^{n+k} q^s \leqslant 0;$$

$$0 \leqslant -\sum_{s=n+1}^{\infty} \ln\bigl(1-q^s x\bigr) - \sum_{s=n+1}^{\infty} q^s x \leqslant \sum_{s=n+1}^{\infty} \frac{q^{2s}x^2}{2(1-q)}.$$

We conclude

$$-\frac{q^{n+2}(1-q^{2k})}{2(1-q^2)(1-q)} \leqslant q^{-n} J - \frac{qx}{1-q} + q[k] \leqslant \frac{x^2 q^{n+2}}{2(1-q^2)(1-q)},$$

which means

$$\left| q^{-n} J - \frac{qx}{1-q} + q[k] \right| < \frac{q^n}{2(1-q)^2}. \tag{4.5}$$

Hence,

$$|J| \leqslant q^n \left(\frac{qx}{1-q} + q[k] + \frac{q^n}{2(1-q)^2} \right) < c_0 q^n, \tag{4.6}$$

where $c_0 = \frac{3}{(1-q)^2}$. Now we show (4.4). Since

$$0 \leqslant e^J - 1 - J = \sum_{n=2}^{\infty} \frac{J^n}{n!} \leqslant \frac{J^2}{2} \sum_{n=2}^{\infty} \frac{c_0^{n-2}}{(n-2)!} \leqslant c_1 J^2, \tag{4.7}$$

where $c_1 = e^{c_0}/2$, by (4.5)–(4.7) we get

$$\begin{aligned} K &\leqslant \bigl| q^{-n}\bigl(1-q^n\bigr)\bigl(e^J - 1 - J\bigr) \bigr| + \left| q^{-n} J - \frac{qx}{1-q} + q[k] \right| + |J| \\ &\ll q^{-n}|J|^2 + q^n + q^n \ll q^{-n}, \end{aligned}$$

which proves Lemma 1. □

Proof of Theorem 3. It follows directly from (1.2) and (2.3) that $M_{n,q}(f,x)$ and $B_{\infty,q}(f,x)$ possess the endpoint interpolation property, in other words,

$$M_{n,q}(f,0) = B_{\infty,q}(f,0) = f(0), \qquad M_{n,q}(f,1) = B_{\infty,q}(f,1) = f(1). \tag{4.8}$$

It follows from (2.1) and Euler's identity (see [5, p. 30]) that

$$\sum_{k=0}^{\infty} m_{nk}(q;x) = \sum_{k=0}^{\infty} m_{\infty k}(q;x) = 1. \tag{4.9}$$

Hence, for all $x \in (0,1)$, by the definitions of $M_{n,q}(f,x)$ and $B_{\infty q}(f,x)$, and by (4.9) we know that

$$\begin{aligned} &\bigl| M_{n,q}(f,x) - B_{\infty,q}(f,x) \bigr| \\ &\quad = \left| \sum_{k=0}^{\infty} \left(f\left(\frac{[k]}{[n+k]} \right) - f(1) \right) m_{nk}(q;x) - \sum_{k=0}^{\infty} \bigl(f\bigl(1-q^k\bigr) - f(1) \bigr) m_{\infty,k}(q;x) \right| \end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{k=1}^{\infty}\left|f\left(\frac{[k]}{[n+k]}\right)-f\left(1-q^k\right)\right|m_{nk}(q;x)\\
&\quad+\sum_{k=0}^{\infty}\left|f\left(1-q^k\right)-f(1)\right|\left|m_{nk}(q;x)-m_{\infty k}(q;x)\right|\\
&=: I_1+I_2. \qquad (4.10)
\end{aligned}$$

First we estimate I_1. Since

$$\left|\frac{[k]}{[n+k]}-\left(1-q^k\right)\right|=\left|\frac{1-q^k}{1-q^{n+k}}-\left(1-q^k\right)\right|=\frac{q^{n+k}(1-q^k)}{1-q^{n+k}}\leqslant q^{n+k}\leqslant q^n,$$

we have

$$I_1\leqslant \omega\left(f,q^n\right)\sum_{k=0}^{\infty}m_{nk}(q;x)=\omega\left(f,q^n\right). \qquad (4.11)$$

Now we estimate I_2. Using (4.2) and the property of modulus of continuity

$$\omega(f,\lambda t)\leqslant(1+\lambda)\omega(f,t),\quad \lambda>0,$$

we obtain

$$\begin{aligned}
I_2&\leqslant \sum_{k=0}^{\infty}\omega\left(f,q^k\right)\left|m_{nk}(q;x)-m_{\infty k}(q;x)\right|\\
&\leqslant \sum_{k=0}^{\infty}\omega\left(f,q^n\right)\left(1+q^{k-n}\right)\left|m_{nk}(q;x)-m_{\infty k}(q;x)\right|\\
&\ll \omega\left(f,q^n\right)\sum_{k=0}^{\infty}m_{\infty,k}(q;x)=\omega\left(f,q^n\right). \qquad (4.12)
\end{aligned}$$

From (4.8), (4.10)–(4.12), we get (2.4).

At last we show that (2.4) is sharp. For each α, $0<\alpha\leqslant 1$, suppose that $f_\alpha(x)$ is a continuous function which is equal to zero in $[0,1-q]$ and $[1-q^2,1]$, equal to $(x-(1-q))^\alpha$ in $[1-q, 1-q+q(1-q)/2]$, and linear in the rest of $[0,1]$. Then

$$\omega(f_\alpha,t)\asymp t^\alpha,$$

and

$$\left\|B_{n,q}(f_\alpha)-B_{\infty,q}(f_\alpha)\right\|=\left(\frac{q^{n+1}(1-q)}{1-q^{n+1}}\right)^\alpha\left\|m_{n1}(q;\cdot)\right\|\asymp q^{\alpha n}.$$

The proof of Theorem 3 is complete. □

In order to prove Theorem 4, we need the following result (see [13]):

Theorem B. *Let the sequence (L_n) of positive linear operators on $C[0,1]$ satisfy the following conditions*:

(A) *The sequence $(L_n(e_2))$ converges to a function $L_\infty(e_2)$ in $C[0,1]$, where $e_i(x)=x^i$, $i=0,1,2$.*

(B) *The sequence* $(L_n(f,x))_{n\geqslant 1}$ *is nonincreasing for any convex function* f *and for any* $x\in[0,1]$.

Then there exists an operator L_∞ *on* $C[0,1]$ *such that* $\|L_n(f)-L_\infty(f)\|\to 0$ *for any* $f\in C[0,1]$. *Furthermore,*

$$\left|L_n(f,x)-L_\infty(f,x)\right|\leqslant c\omega_2\left(f,\sqrt{\lambda_n(x)}\right), \tag{4.13}$$

where $\lambda_n(x)=L_n(e_2,x)-L_\infty(e_2,x)$, *c is a constant dependent only on* $\|L_1(e_0)\|$.

Proof of Theorem 4. From [12], we know that the q-MKZ operators satisfy condition (B). From Theorem 3 we know that for $q\in(0,1)$,

$$M_{n,q}(f,x)\rightrightarrows B_{\infty,q}(f,x)\quad \left[x\in[0,1];\, n\to\infty\right].$$

Also,

$$\begin{aligned}
0\leqslant \lambda_n(x)&=M_{n,q}\left(t^2,x\right)-M_{\infty,q}\left(t^2,x\right)\\
&=\frac{x(1-x)}{[n+1]}\left(1-\frac{q^{n+2}[n]x}{[n+2]}{}_2\phi_1\left(q,q^2;q^{n+3};q,q^{n+1}x\right)\right)-(1-q)x(1-x)\\
&\leqslant \frac{x(1-x)}{[n+1]}-(1-q)x(1-x)=\frac{q^{n+1}(1-q)}{1-q^{n+1}}x(1-x)\\
&\leqslant q^{n+1}x(1-x)\leqslant q^n,
\end{aligned} \tag{4.14}$$

and

$$\sup_{0<q<1}\left|M_{n,q}\left(t^2,x\right)-M_{\infty,q}\left(t^2,x\right)\right|\leqslant \sup_{0<q<1}\frac{q^{n+1}(1-q)}{1-q^{n+1}}x(1-x)=\frac{x(1-x)}{n+1}.$$

Since we know that

$$\left|M_{n,1}\left(t^2,x\right)-x^2\right|\leqslant\frac{x(1-x)}{n+1},$$

we conclude that

$$\sup_{0<q\leqslant 1}\left|M_{n,q}\left(t^2,x\right)-M_{\infty,q}\left(t^2,x\right)\right|\leqslant\frac{x(1-x)}{n+1}\leqslant 1/n. \tag{4.15}$$

Theorem 4 follows from (4.14), (4.15), and Theorem B. □

5. Proof of Theorem 5

Proof of Theorem 5. It follows directly from the definitions of $M_{n,q}(f,x)$, $B_{\infty,q}(f,x)$ and $V_q(f,x)$ that

$$\begin{aligned}
&M_{n,q}(f,0)=B_{\infty,q}(f,0)=f(0),\qquad M_{n,q}(f,1)=B_{\infty,q}(f,1)=f(1),\\
&V_q(f,0)=V_q(f,1)=0.
\end{aligned}$$

Then it suffices to show that

$$\lim_{n\to\infty}\frac{[n]}{q^n}\left(M_{n,q}(f,x)-B_{\infty,q}(f,x)\right)=L_q(f,x)$$

uniformly in $x\in(0,1)$.

For $f \in C^1[0,1]$, there exists a constant $M > 1$ such that $|f(x)| \leqslant M$, $|f'(x)| \leqslant M$. Let $\varepsilon > 0$ be given. Let $\delta \in (0,1)$ be chosen in such a way that $|f'(x_1) - f'(x_2)| \leqslant (1-q)\varepsilon/6$ whenever $|x_1 - x_2| \leqslant \delta$. Let R be a positive integer satisfying the condition $q^R < \delta$. We estimate the difference

$$\Delta := \left|\frac{[n]}{q^n}\big(M_{n,q}(f,x) - B_{\infty,q}(f,x)\big) - V_q(f,x)\right|$$

for $n > R$ and $x \in (0,1)$. Using the fact that $m_{\infty k}(q;x) = \frac{x}{(1-q)[k]} m_{\infty k-1}(q;x)$ and the definition of $V_q(f,x)$, we get for $x \in (0,1)$,

$$\begin{aligned}
V_q(f,x) &= \sum_{k=1}^{\infty} \frac{q^k[k] f(1-q^{k-1})}{q^{k-1}-q^k} m_{\infty k}(q;x) \\
&\quad - \sum_{k=1}^{\infty} \left(\frac{q^k[k] f(1-q^k)}{q^{k-1}-q^k} - q^k[k] f'\big(1-q^k\big)\right) m_{\infty k}(q;x) \\
&= \sum_{k=1}^{\infty} \frac{qxf(1-q^{k-1})}{(1-q)^2} m_{\infty k-1}(q;x) \\
&\quad - \sum_{k=1}^{\infty} \left(\frac{q[k] f(1-q^k)}{1-q} - q^k[k] f'\big(1-q^k\big)\right) m_{\infty k}(q;x) \\
&= \sum_{k=0}^{\infty} \left(\left(\frac{qx}{(1-q)^2} - \frac{q[k]}{1-q}\right) f\big(1-q^k\big) + q^k[k] f'\big(1-q^k\big)\right) m_{\infty,k}(q;x). \qquad (5.1)
\end{aligned}$$

By (1.2), (2.3) and (5.1), we have

$$\begin{aligned}
\Delta &= \left|\frac{[n]}{q^n}\big(M_{n,q}(f,x) - B_{\infty,q}(f,x)\big) - V_q(f,x)\right| \\
&= \Bigg|\sum_{k=0}^{\infty} \left(\frac{[n]}{q^n}\big(f\big([k]/[n+k]\big) - f\big(1-q^k\big)\big) - q^k[k] f'\big(1-q^k\big)\right) m_{nk}(q;x) \\
&\quad + \sum_{k=0}^{\infty} f\big(1-q^k\big)\left(\frac{[n]}{q^n}\big(m_{nk}(q;x) - m_{\infty k}(q;x)\big) - \left(\frac{qx}{(1-q)^2} - \frac{q[k]}{1-q}\right) m_{\infty k}(q;x)\right) \\
&\quad + \sum_{k=0}^{\infty} q^k[k] f'\big(1-q^k\big)\big(m_{nk}(q;x) - m_{\infty k}(q;x)\big)\Bigg| \\
&\leqslant \sum_{k=0}^{\infty} \left|\frac{[n]}{q^n}\big(f\big([k]/[n+k]\big) - f\big(1-q^k\big)\big) - q^k[k] f'\big(1-q^k\big)\right| m_{nk}(q;x) \\
&\quad + \sum_{k=0}^{\infty} \big|f\big(1-q^k\big)\big| \left|\frac{[n]}{q^n}\big(m_{nk}(q;x) - m_{\infty k}(q;x)\big) - \left(\frac{qx}{(1-q)^2} - \frac{q[k]}{1-q}\right) m_{\infty k}(q;x)\right| \\
&\quad + \sum_{k=0}^{\infty} q^k[k] \big|f'\big(1-q^k\big)\big| \big|m_{nk}(q;x) - m_{\infty k}(q;x)\big| =: I_1 + I_2 + I_3. \qquad (5.2)
\end{aligned}$$

First we estimate I_2 and I_3. Using (4.2) and (4.3), we obtain

$$I_2 \leqslant \sum_{k=0}^{\infty} cMq^n m_{\infty k}(q;x) \leqslant cMq^n, \tag{5.3}$$

$$I_3 \leqslant \sum_{k=0}^{\infty} \frac{cM}{1-q} q^n m_{\infty k}(q;x) \leqslant \frac{cM}{1-q} q^n. \tag{5.4}$$

Now we estimate I_1. Since

$$\begin{aligned}
&\left|\frac{[n]}{q^n}\left(f\left(\frac{[k]}{[n+k]}\right) - f\left(1-q^k\right)\right) - q^k[k]f'\left(1-q^k\right)\right| \\
&\quad = q^k[k]\left|\frac{1-q^n}{1-q^{n+k}} f'(\xi_k) - f'\left(1-q^k\right)\right| \quad \left(\xi_k \in \left(1-q^k, \frac{[k]}{[n+k]}\right)\right) \\
&\quad \leqslant q^k[k]\left|f'(\xi_k) - f'\left(1-q^k\right)\right| + \frac{q^n(1-q^k)}{1-q^{n+k}}\left|f'(\xi_k)\right| \\
&\quad \leqslant \frac{1}{1-q}\varepsilon + Mq^n,
\end{aligned}$$

we get

$$I_1 \leqslant \sum_{k=0}^{\infty}\left(\frac{1}{1-q}\varepsilon + Mq^n\right) m_{nk}(q;x) \leqslant \varepsilon/6 + Mq^n. \tag{5.5}$$

Thus, by (5.2)–(5.5) we obtain $\Delta < \varepsilon$ for n sufficiently large. The proof of Theorem 5 is complete. $\square$

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